

exercise : Compatible operator A and B

18

Q. $|a\rangle \xrightarrow{\text{A measurement}} |?\rangle \xrightarrow{\text{B measurement}} |?\rangle \xrightarrow{\text{A meas.}} |?\rangle$

A. i) non degenerate : obvious!

$|a\rangle \xrightarrow{A} |a, b\rangle \xrightarrow{B} |a, b\rangle \xrightarrow{A} |a, b\rangle$

ii) degenerate (n-fold)

$|a\rangle \xrightarrow{A} \boxed{?} \xrightarrow{B} \boxed{?} \xrightarrow{A} \boxed{?}$
 $\nwarrow \quad \uparrow \quad \nwarrow$
 $\sum_{i=1}^n c_a^{(i)} |a, b^{(i)}\rangle \quad |a, b^{(j)}\rangle \quad |a, b^{(j)}\rangle$

A and B do not interfere ! ("compatible")

: B between A's does not change the measured result "a".
 (non-deg.)

(4) Incompatible observables

if $[A, B] \neq 0$, they do interfere with each other.

there are no simultaneous eigenkets
 (in general)

✓ exception : in some case, "subspace" has the same eigenkets.

ex. S-orbital : $(l=0)$

it's proportional to p_z .

→ this is a simultaneous eigenket

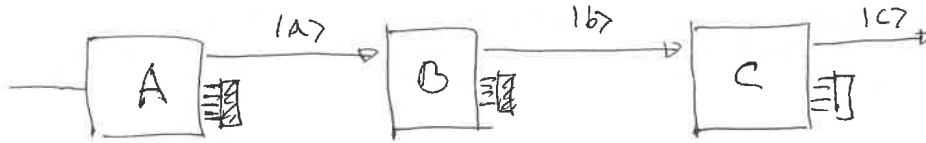
of L_x and L_y .

Interference between A and B $\parallel [A, B] \neq 0$

19

Compare two experiments :

i)



ii)



• What's the difference? - Both gives "|c>" !

The Probability to get |c> is different from one to another.

$$i) P_1 = |\langle c|b\rangle|^2 |\langle b|a\rangle|^2$$

$$ii) P_2 = |\langle c|a\rangle|^2$$

* What if we "sum up all b" in i) - exp by repeating exp.

$$\Rightarrow \sum_b |\langle c|b\rangle|^2 |\langle b|a\rangle|^2 = \sum_b \langle c|b\rangle \langle b|a\rangle \langle a|b\rangle \langle b|c\rangle$$

Still, it's different from P_2 !

$$P_2 = |\langle c|a\rangle|^2 = \left| \sum_{b'} \langle c|b'\rangle \langle b'|a\rangle \right|^2$$

$$= \sum_{b', b''} \langle c|b'\rangle \langle b'|a\rangle \langle a|b''\rangle \langle b''|c\rangle$$

↑ This is the "Quantum" nature.

Note: $P_1 = P_2$ when $\left(\begin{array}{l} [A, B] = 0 \\ \text{or} \\ [B, C] = 0 \end{array} \right)$ and "nondegenerate" !

$|a\rangle = |b\rangle = |a, b\rangle$
 $|b\rangle = |c\rangle = |b, c\rangle$

(5) Example: The Uncertainty Relation (Incompatible), 20.

def. $\Delta A \equiv A - \langle A \rangle$: an operator to measure the fluctuation. (derivation)

dispersion (variance)

$$\langle (\Delta A)^2 \rangle = \langle (A^2 - 2A\langle A \rangle + \langle A \rangle^2) \rangle = \underline{\langle A^2 \rangle - \langle A \rangle^2}$$

uncertainty relation: $\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq \frac{1}{4} |\langle [A, B] \rangle|^2$

Let's prove this.

proof.

Since $\langle (\Delta A)^2 \rangle = \langle \alpha | \Delta A \cdot \Delta A | \alpha \rangle$, likewise for B.
 (Some ket. (arb.))

• The Schwarz inequality gives

$$\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq |\langle \Delta A \Delta B \rangle|^2$$

$$\begin{aligned} \langle \alpha | \alpha \rangle \langle \beta | \beta \rangle &\geq |\langle \alpha | \beta \rangle|^2 \\ \text{(vector formula)} \\ &\therefore |\vec{a}|^2 |\vec{b}|^2 \geq |\vec{a} \cdot \vec{b}|^2 \end{aligned}$$

$$\Delta A \Delta B = \frac{1}{2} [\Delta A, \Delta B] + \frac{1}{2} \{ \Delta A, \Delta B \}$$

$$\underline{\hspace{10em}} = [A, B] \quad \parallel \begin{matrix} \langle A \rangle, \langle B \rangle \\ \text{are just numbers} \end{matrix}$$

where $[A, B]$: anti-Hermitian

$\langle [A, B] \rangle$: pure imaginary

$$[A, B]^{\dagger} = (AB - BA)^{\dagger} = BA - AB = -[A, B]$$

$\{ \Delta A, \Delta B \}$: Hermitian

$\langle \{ \Delta A, \Delta B \} \rangle$: real

$$\parallel \langle \Delta A \Delta B \rangle = 0i + 0$$

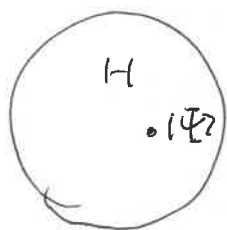
$$\text{Thus } |\langle \Delta A \Delta B \rangle|^2 = \frac{1}{4} |\langle [A, B] \rangle|^2 + \frac{1}{4} |\langle \{ \Delta A, \Delta B \} \rangle|^2$$

$$\geq \frac{1}{4} |\langle [A, B] \rangle|^2$$

1.6 Change of basis

(1) Transformation operator.

There are multiple choices of base kets to span the same H -space.



$$|\Psi\rangle = \sum_a c_a |a\rangle$$

$$|\Psi\rangle = \sum_b c_b |b\rangle$$

$$|\Psi\rangle = \sum_c c_c |c\rangle \quad \dots \text{many!}$$

$$|a\rangle \xleftarrow{U, U^\dagger} |b\rangle$$

How ARE they related to each other?

• unitary transformation.

Theorem

There " U " such that $|b^{(l)}\rangle = U |a^{(l)}\rangle \quad l=1 \dots N$
 exists $\uparrow \quad \uparrow$
 [orthonormal, completeness] \uparrow
 dim. of H -space

$$\| \quad U : \text{unitary operator} \quad \begin{array}{l} \text{def.} \\ \text{iff.} \end{array} \quad \begin{array}{l} U^\dagger U = I \\ U U^\dagger = I \end{array} \quad \star$$

\hookrightarrow An obvious construction of U :

$$U = \sum_k |b^{(k)}\rangle \langle a^{(k)}|. \quad \underbrace{|a\rangle \xrightleftharpoons[U^\dagger]{U} |b\rangle}$$

check: $U^\dagger U = \sum_{k,l} |a^{(k)}\rangle \langle b^{(k)}| \underbrace{|b^{(l)}\rangle \langle b^{(l)}|}_{\delta_{kl}} \langle a^{(l)}| = I$

$$U U^\dagger = \sum_{k,l} |b^{(k)}\rangle \langle a^{(k)}| \underbrace{|a^{(l)}\rangle \langle a^{(l)}|}_{\delta_{kl}} \langle b^{(l)}| = I$$

(2) Transformation Matrix

22

$$\{ |a\rangle \} \xrightarrow{\boxed{U} : \text{unitary op.}} \{ |b\rangle \}$$

(old) (new)

$$U = \sum_k |b^{(k)}\rangle \langle a^{(k)}|$$

• matrix representation of U

$$\begin{aligned} \langle a^{(k)} | U | a^{(l)} \rangle &= \sum_{k'} \langle a^{(k)} | b^{(k')} \rangle \langle a^{(k')} | a^{(l)} \rangle \\ &\quad \hookrightarrow \delta_{k'l} \\ &= \langle a^{(k)} | b^{(l)} \rangle \\ &\equiv U_{kl} : \text{transformation matrix.} \end{aligned}$$

• transformation of $|\alpha\rangle$ in the old basis

i.e. $|\alpha\rangle = \sum_{a'} |\alpha\rangle \langle a' | \alpha \rangle \rightarrow \text{coeff. } \underline{\langle a' | \alpha \rangle}$

to the one in the new basis:

$$|\alpha\rangle = \sum_{b'} |b'\rangle \langle b' | \alpha \rangle \rightarrow \text{coeff. } \underline{\langle b' | \alpha \rangle}$$

Q.

$$\begin{pmatrix} \langle a' | \alpha \rangle \end{pmatrix} \xrightarrow{?} \begin{pmatrix} \langle b' | \alpha \rangle \end{pmatrix} : \text{matrix representation.}$$

$$\langle b^{(k)} | \alpha \rangle = \sum_l \underbrace{\langle b^{(k)} | a^{(l)} \rangle}_{= U_{lk}^*} \langle a^{(l)} | \alpha \rangle \quad \parallel \quad |b^{(k)}\rangle = U |a^{(k)}\rangle$$

$$= \sum_l \langle a^{(k)} | U^\dagger | a^{(l)} \rangle \langle a^{(l)} | \alpha \rangle$$

In matrix representation, $|\alpha\rangle$ in a-basis $\xrightarrow{[U^\dagger] \text{ matrix.}}$ $|\alpha\rangle$ in b-basis.

$$\begin{pmatrix} \vdots \\ \langle b^{(k)} | \alpha \rangle \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots & \vdots & \vdots \\ \vdots & \boxed{U^\dagger} & \vdots \\ \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} \langle a^{(1)} | \alpha \rangle \\ \vdots \\ \langle a^{(l)} | \alpha \rangle \end{pmatrix}$$

$= U_{lk}^*$

it's a dagger!
not " U "!

Q. What about the matrix rep. of an operator 23.

$$\begin{pmatrix} \langle a^{(1)} | \cdot | a^{(1)} \rangle \\ \vdots \\ \langle a^{(n)} | \cdot | a^{(n)} \rangle \end{pmatrix} \xrightarrow[\text{where } |b\rangle = U|a\rangle]{?} \begin{pmatrix} \langle b^{(1)} | X | b^{(1)} \rangle \\ \vdots \\ \langle b^{(n)} | X | b^{(n)} \rangle \end{pmatrix}$$

$$\begin{aligned} \langle b^{(k)} | X | b^{(k)} \rangle &= \langle b^{(k)} | \cdot \sum_n |a^{(n)}\rangle \langle a^{(n)}| \cdot X \cdot \sum_n |a^{(n)}\rangle \langle a^{(n)}| \cdot | b^{(k)} \rangle \\ &= \sum_{m,n} \langle b^{(k)} | a^{(m)} \rangle \langle a^{(n)} | X | a^{(n)} \rangle \langle a^{(m)} | b^{(k)} \rangle \end{aligned}$$

$$\Rightarrow X' = U^\dagger X U$$

Similarity transformation.

property. $\text{tr}(X) = \text{tr}(X')$: trace is conserved

\rightarrow an invariant of unitary-matrix transformation.
 \uparrow
 similarity

It's just a ^{math.} property of "trace".

$$\text{tr}(U^\dagger X U) = \text{tr}(X U U^\dagger) = \text{tr}(X)$$

likewise,

$$\text{tr}(|a'\rangle \langle a'|) = \delta_{a'a'}$$

$$\text{tr}(|b'\rangle \langle a'|) = \langle a' | b' \rangle$$

(3) Diagonalization.

24.

- How to solve $B|b\rangle = b|b\rangle$? $\parallel \begin{pmatrix} |b\rangle \\ b \end{pmatrix}$ unknowns.

- What do we know? Say, we know some base kets to compute $\langle a_i | B | a_j \rangle$.

$$\rightarrow \langle a_i | \cdot (\underbrace{B|b\rangle}_{I = \sum_j |a_j\rangle\langle a_j|}) = b \langle a_i | b \rangle$$

$$\Rightarrow \sum_j \underbrace{\langle a_i | B | a_j \rangle}_{\substack{\downarrow \\ \text{matrix}}} \underbrace{\langle a_j | b \rangle}_{\substack{\downarrow \\ \text{col. vector}}} = b \underbrace{\langle a_i | b \rangle}_{\substack{\downarrow \\ \text{col. vector}}}$$

$$\begin{pmatrix} \ddots & & \\ & B_{ij} & \\ & \equiv \langle a_i | B | a_j \rangle & \\ & & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ \langle a_j | b \rangle \\ \vdots \end{pmatrix} = b \begin{pmatrix} \vdots \\ \langle a_j | b \rangle \\ \vdots \end{pmatrix}$$

\therefore linear equation for the eigenproblem.

$$\rightarrow \text{to find } b : \det(\tilde{B} - \lambda \tilde{I}) = 0$$

and just do what you do for the matrix diagonalization.

The most important thing:

To find (relevant) base kets
(good)

to express your operator B .

ex.

$$\hat{S}_x = \frac{\hbar}{2} [| \uparrow \rangle \langle \downarrow | + | \downarrow \rangle \langle \uparrow |]$$

in $\{ | \uparrow \rangle, | \downarrow \rangle \}$ basis.

(4) Unitary Equivalent Observables

25

$$A \xrightarrow[\substack{\text{unitary transform.} \\ U|a\rangle = |b\rangle}]{U^{-1}} U A U^+$$

Same eigenvalues (spectrum) but different eigenket.

A and $U A U^+$ are unitary equivalent observables.

proof

$$A |a^{(e)}\rangle = a^{(e)} |a^{(e)}\rangle$$

$$U A U^+ U |a^{(e)}\rangle = a^{(e)} U |a^{(e)}\rangle$$

\parallel
 I

$$\Rightarrow (U A U^+) |b^{(e)}\rangle = a^{(e)} |b^{(e)}\rangle$$

unitary transformation does not change spectrum!

ex. S_x, S_y, S_z (spin $-\frac{1}{2}$)

These are related through the unitary transformations.

\rightarrow eigenvalues are: $+\frac{\hbar}{2}, -\frac{\hbar}{2}$

for all of S_x, S_y, S_z

1.6 Position, Momentum, and Translation.

26

(1) continuous spectra

$$\Sigma \rightarrow \int$$

$$\delta_{a;a} \rightarrow \delta(\tilde{a} - \tilde{a};)$$

ex) completeness rel.

$$\int d\tilde{z} |\tilde{z}\rangle \langle \tilde{z}| = 1.$$

$$\text{ex) } \langle \tilde{z}' | \tilde{z} \rangle = \delta(\tilde{z}' - \tilde{z})$$

$$\text{ex) } \langle \beta | \alpha \rangle = \int d\tilde{z} \langle \beta | \tilde{z} \rangle \langle \tilde{z} | \alpha \rangle$$

Notation Note !!!

$$\text{"tilde"} \tilde{z} |\tilde{z}\rangle = \tilde{z} |\tilde{z}\rangle \quad (\text{on } \tilde{z})$$

\uparrow operator \rightarrow c-number

$$A |\tilde{z}\rangle = \tilde{z} |\tilde{z}\rangle$$

\uparrow capital: an operator

$$A |a\rangle = a |a\rangle$$

\uparrow lower cap.
: c-number

In Sakurai,

primed symbols: c-number.

unprimed: an operator.

$$\text{ex. } x |x'\rangle = x' |x'\rangle$$

How to read

$$\tilde{z} |\tilde{z}\rangle = \tilde{z} |\tilde{z}\rangle :$$

ket $|\tilde{z}\rangle$ is an eigenvector of operator \tilde{z}
with eigenvalue \tilde{z} .

(2) Position Eigenkets and Position Measurements.

$$x |x\rangle = x |x\rangle \quad \rightarrow \text{position eigenket. (localized at } x)$$

\uparrow position: an eigenvalue of x

x -operator measuring a position from $|x\rangle$

• completeness relation

$$\int dx |x\rangle \langle x| = 1. \quad \Rightarrow \text{definite integral}$$

\rightarrow over all space.

- The state ket for an arbitrary physical state.

27

$$|a\rangle = \int_{-\infty}^{\infty} dx |x\rangle \langle x|a\rangle$$

← continuum version of
 $|a\rangle = \sum_a |a\rangle \langle a|a\rangle$



probability to find $|a\rangle$ in the narrow interval around x

$$= |\langle x|a\rangle|^2 dx$$

↑
 probability density.

- In 3D, $|\vec{x}\rangle \equiv (x, y, z)$

$$\tilde{x}|\vec{x}\rangle = x|\vec{x}\rangle, \quad \tilde{y}|\vec{x}\rangle = y|\vec{x}\rangle, \quad \tilde{z}|\vec{x}\rangle = z|\vec{x}\rangle$$

⏟
 "simultaneous" eigenket!

$$\leftarrow [\tilde{x}_i, \tilde{x}_j] = 0.$$

(3) Translation operator.

$$|\vec{x}\rangle \xrightarrow{J(\delta\vec{x})} |\vec{x} + \delta\vec{x}\rangle$$

↗ make translation from \vec{x} to $\vec{x} + \delta\vec{x}$

"infinitesimal"

$$J(\delta\vec{x})|\vec{x}\rangle = |\vec{x} + \delta\vec{x}\rangle$$

meaning: $\delta\vec{x}$ is too small
 to change anything else.